June 19, 20, and 21, 2023. The area methods for G8, G9, G10.

Theorem 1 (the co-side theorem)**.** *Let M be the intersection of two lines AB and PQ. Then* $\overline{PM} : \overline{QM} = \Delta PAB : \Delta QAB$.

Theorem 2 (the co-angle theorem)**.** *If $\angle ABC = \angle XYZ$ or $\angle ABC + \angle XYZ = 180°$, we have* $\Delta ABC / \Delta XYZ = \overline{AB} \cdot \overline{BC} / \overline{XY} \cdot \overline{YZ}$.

Theorem 3. *Let R be a point on line PQ. Then for any two points A and B,*

$$\Delta RAB = \frac{\overline{PR}}{\overline{PQ}} \Delta QAB + \frac{\overline{RQ}}{\overline{PQ}} \Delta PAB.$$

Problem 1. Prove Menelaus' theorem and Ceva's theorem using the co-side theorem.

Problem 2. Prove Pappus's theorem using the co-side theorem. Pappus's theorem: If the vertices of a hexagon fall alternately on two lines, the intersections of opposite sides are collinear.

Problem 3 (Easy)**.** $ABCD$ is a convex quadrilateral with E, F being the points of trisection on BC and G, H being the points of trisection on DA. Prove that $[ABCD] = 3[EFGH]$

Problem 4 (Moderate)**.** $ABCD$ is a convex quadrilateral with E, F being the points of trisection on BC, G, H being the points of trisection on DA, I, J being the points of trisection on AB and K, L being the points of trisection on CD. The connectors of the corresponding trisection points on the opposite sides of $ABCD$ form quadrilateral $MNOP$. Prove that $[ABCD] = 9[MNOP]$.

Problem 5 (Moderate, Poland MO 1965)**.** All three diagonals AD, BE, CF of a convex hexagon $ABCDEF$ bisect the area. Prove that AD, BE, CF concur.

Problem 6 (Moderate)**.** Point I is the incenter of triangle ABC. Prove that

$$\frac{\overline{AI}^2}{\overline{AB} \cdot \overline{AC}} + \frac{\overline{BI}^2}{\overline{BC} \cdot \overline{BA}} + \frac{\overline{CI}^2}{\overline{CA} \cdot \overline{CB}} = 1.$$

Problem 7 (Generalized Butterfly)**.** A, B, C, D, E, F are six points on a circle. $M = AB \cap CD$; $N = AB \cap EF$; $G = AB \cap CF$; $H = AB \cap DE$. Show that $\overline{MG} \cdot \overline{BH} \cdot \overline{AN} = \overline{AG} \cdot \overline{NH} \cdot \overline{MB}$.

Problem 8 (Moderate, Cheng Zhong)**.** G is the centroid of triangle ABC. A_1, B_1, C_1 are the intersection of rays AG, BG, CG with the circumcircle of ABC. Suppose G is the midpoint of AA_1. Extend A_1B, A_1C and intersect line B_1C_1 at C_2, B_2 respectively. Prove that $\overline{B_1B_2} / \overline{C_1C_2} = \left(\overline{AC} / \overline{AB}\right)^2$.

Problem 9 (Difficult)**.** In a cyclic convex quadrilateral $ABCD$, E is the intersection of rays AD, BC, and F the intersection of rays BA, CD, and G the intersection of AC, BD. Prove that $\cos \angle ABC / \overline{BG} = \cos \angle BAC / \overline{AE} + \cos \angle BCA / \overline{CF}$.