June 22 and 23, 2023. Convexity for G1, G2, G4.

Theorem 1 (Helly's theorem)**.** For every n and d with $n \geq d+1$, and for every n convex subsets $F_1, \ldots, F_n$ of $\mathbb{R}^d$, if every $d+1$ of them have non-empty intersection, then $F_1 \cap \cdots \cap F_n \neq \varnothing$.

Theorem 2 (Radon's theorem)**.** Any set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls intersect.

Problem 1 (Centerpoint)**.** Let X be an n-point set in $\mathbb{R}^d$. A point $x \in \mathbb{R}^d$ is called a *centerpoint* of X if each closed half-space containing x contains at least $\frac{n}{d+1}$ points of X. Each finit point set in $\mathbb{R}^d$ has at least one centerpoint.

Problem 2. Let $K_1, \ldots, K_n$ be closed intervals parallel to the y-axis. Assume that for any $|I| \leq d+2$ there exists a polynomial of degree at most d, the graph of which intersects all K_i where $i \in I$. Show that there exists a polynomial of degree at most d, the graph of which intersects all the intervals $K_1, \ldots, K_n$.

Problem 3 (The art of mathematics, Bollobas, Problem 127)**.** Let C be a compact convex body in $\mathbb{R}^n$ with non-empty interior. A maximal interval $[u, v]$ contained in C is a chord of C. Show that C contains a point c such that every chord $[u, v]$ through c satisfies $|c - u| \leq \frac{n}{n+1} |v - u|$.

Problem 4 (Yaglom & Boltyanskii's Convex Figures, Problem 18)**.**

(a) Given a bounded curve in the plane (possibly not connected), prove there exists a point O such that every line through O cuts off at least $1/3$ of the length of the curve.

(b) Given a bounded figure in the plane (possibly not connected), prove there exists a point O such that every line through O cuts off at least $1/3$ of the area.

Problem 5 (Sallee, Buchman–Valentine)**.**

(a) Prove that a compact convex set of width 1 contains a segment of length 1 of every direction.

(b) Let $C_1, \ldots, C_n$ be closed convex sets in the plane, $n \geq 3$, such that the intersection of every 3 of them has width at least 1. Prove that $\cap_{i=1}^n C_i$ has width at least 1.

Problem 6. Let $K \subset \mathbb{R}^d$ be a convex set and let $C_1, \ldots, C_n \subset \mathbb{R}^d$, $n \geq d+1$, be convex sets such that the intersection of every $d+1$ of them contains a translated copy of K. Prove that then the intersection of all the sets C_i also contains a translated copy of K.

Problem 7 (Kirchberger's theorem)**.** Given disjoint finite sets A and B in $\mathbb{R}^d$. Suppose that for any set $S \subset \mathbb{R}^d$ with $|S| \leq d+2$, there exists a hyperplane which strictly separates $S \cap A$ and $S \cap B$. Then there exists a hyperplane which strictly separates the sets $S \cap A$ and $S \cap B$. Then there exists a hyperplane which strictly separates A and B.

Problem 8. Prove that if the intersection of each 4 or fewer among convex sets $C_1, \ldots, C_n \subset \mathbb{R}^2$ contains a ray then $\cap_{i=1}^n C_i$ also contains a ray. Show that the number 4 cannot be replaced by 3.

Problem 9 (H. Jung's theorem)**.** Each finite set $X \subset \mathbb{R}^2$ of diameter at most 1 is contained in some disc of radius $1/\sqrt{3}$.

Problem 10. For a set $X \subset \mathbb{R}^2$ and a point $x \in X$, denote by $V(x)$ the set of all points $y \in X$ that can "see" x, i.e., points such that the segment xy is contained in X. The kernel of X is defined as the set of all points $x \in X$ such that $V(x) = X$. A set with a nonempty kernel is called star-shaped.

(a) Prove that the kernel of any set is convex.

(b) Construct a nonempty set $X \subset \mathbb{R}^2$ such that each of its finite subsets can be seen from some point of X but X is not star-shaped.