June 26 and 29, 2023. Three tricks in geometry for G8, G9, G10.

Theorem 1 (1st trick)**.** Points C_0 and A_0 are chosen on the sides AB and BC of the triangle ABC respectively. Point B_1 is the midpoint of the arc ABC of the circumcircle of the triangle ABC. Prove that $\overline{AC_0} = \overline{CA_0}$ if and only if A_0, C_0, B_1, B lie on a circle.

Theorem 2 (2nd trick)**.** Points C_0 and A_0 are on the sides AB and BC of the triangle ABC respectively. Point I is the incenter of ABC. Point J is the midpoint of the arc AC of the circumcircle of ABC. Prove that

(a) the circumcircle of A_0BC_0 passes through I if and only if $\overline{AC_0} + \overline{CA_0} = \overline{AC}$.
(b) the circumcircle of A_0BC_0 passes through J if and only if $\overline{BC_0} + \overline{BA_0} = \overline{BA} + \overline{BC}$.

Theorem 3 (3rd trick)**.** Points X and Y move at constant speed (not necessarily equal) along two straight lines intersecting at O. Prove that the circumcircle of XYO passes through two fixed points O and Z, where Z is the center of the spiral similarity between the locations of X and Y.

Theorem 4 (Miquel's theorem)**.** Given four lines l_1, l_2, l_3, l_4 (in general position). Denote by ω_1 the circumcircle of the triangle formed by l_2, l_3, l_4. Analogously define $\omega_2, \omega_3, \omega_4$. Prove theses circles pass through the same point.

Problem 1 (Romanian Masters in Mathematics 2015 Day 2 Problem 4)**.** Let ABC be a triangle, and let D be the point where the incircle meets the side BC. Let J_b and J_c be the incenters of the triangles ABD and ACD, respectively. Prove that the circumcenter of the triangle AJ_bJ_c lies on the angle bisector of $\angle BAC$.

Problem 2 (All-Russian Olympiad 2005 Grade 11 Day 1 Problem 3)**.** Let A', B', C' be points where the excircles touch the corresponding sides of the triangle ABC. Circumcircles of the triangles $A'B'C$, $AB'C'$, $A'BC'$ intersect the circumcircle of ABC at points $C_1 \neq C, A_1 \neq A, B_1 \neq B$ respectively. Prove that the triangle $A_1B_1C_1$ is similar to the triangle formed by the points where the incircle of ABC touches its sides.

Problem 3 (Tournament of Towns 1999 Grade 10-11 Problem 4b)**.** Let C_0 and A_0 be points on the sides BA and BC of the triangle ABC respectively, and let the points M and M_0 be the midpoints of segments AC and A_0C_0. Prove that if $AC_0 = CA_0$, then the line MM_0 is parallel to the bisector of $\angle ABC$.

Problem 4 (IMO 2013 Day 1 Problem 3)**.** Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcenter of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

Problem 5 (All-Russian Olympiad 2012 Grade 9 Day 2 Problem 2)**.** The points A_1, B_1, C_1 lie on the sides sides BC, AC and AB of the triangle ABC respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let I_A, I_B, I_C be the incenters of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the circumcenter of triangle $I_AI_BI_C$ is the incenter of triangle ABC.

Problem 6 (All-Russian Olympiad 2012 Grade 11 Day 2 Problem 2)**.** The points A_1, B_1, C_1 lie on the sides BC, CA and AB of the triangle ABC respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let O_A, O_B and O_C be the circumcenters of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the incenter of triangle $O_AO_BO_C$ is the incenter of triangle ABC too.

Problem 7 (IMO Shortlist 2012 G6). Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

Problem 8 (All-Russian Olympiad 2011 Grade 11 Day 2 Problem 4). Let N be the midpoint of arc ABC of the circumcircle of triangle ABC, let M be the midpoint of AC and let I_1, I_2 be the incenters of triangles ABM and CBM. Prove that points I_1, I_2, B, N lie on a circle.

Problem 9 (IMO 1985 Day 2 Problem 5). A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90°$.

Problem 10 (All-Russian Olympiad 2000 Grade 10 Day 1 Problem 3). In an acute scalene triangle ABC the bisector of the acute angle between the altitudes AA_1 and CC_1 meets the sides AB and BC at P and Q respectively. The bisector of the angle B intersects the segment joining the orthocenter of ABC and the midpoint of AC at point R. Prove that P, B, Q, R lie on a circle.

Problem 11 (IMO Shortlist 2006 G9). Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

Problem 12 (All-Russian Olympiad 2001 Grade 10 Day 2 Problem 3). Points A_1, B_1, C_1 inside an acute-angled triangle ABC are selected on the altitudes from A, B, C respectively so that the sum of the areas of triangles ABC_1, BCA_1, and CAB_1 is equal to the area of triangle ABC. Prove that the circumcircle of triangle $A_1B_1C_1$ passes through the orthocenter H of triangle ABC.

Problem 13 (Iranian National Mathematical Olympiad 1997 Round 4 Problem 4). Point E is chosen on the arc AC of the circumcircle Ω of triangle ABC. Let I_a and I_c be the incenters of triangles AEB and CEB, let Ω' be the circle tangent to AB, CB and Ω. The circles Ω and Ω' meet at T_b. Prove that I_a, I_c, E, T_b lie on a circle.

Problem 14 (IMO 2005 Day 2 Problem 5). Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

Problem 15 (All-Russian Olympiad 2014 Grade 10 Day 1 Problem 4). Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incenter of the triangle AMK and Q be the K-excentre of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.